Summer HSSP Week 1 Homework

Lane Gunderman, Victor Lopez, James Rowan

July 9, 2014

Questions

Chapter 1

Homework Questions

These are the questions that should be turned in as homework. As mentioned in class, do as many as you like. In general (*) marked questions are straight forward, (**) marked questions require a little bit of cleverness, (***) requires a fair amount of cleverness and or care, while (****) are questions that would challenge the teachers to some degree, and lastly (*****) marked questions are unsolved questions in math, which are meant merely to challenge the most advanced students and introduce everyone to really tough problems. Generally speaking, you should be at the point of fully understanding the material if you are able to do most of the (*) questions in a section and some of the (**) questions. Anything beyond that is for extra practice, and may be so accordingly rewarded at the end of the class series. We'll see.

1.0.1 Questions for chapter 2

Categorize the following numbers:
(*)1. $2;\ -1;\ 0;\ \sqrt{2};\ \pi;\ \frac{3}{2}$

(*)2. $\frac{16384}{32};\ 0.\bar{7};\ 11$

(**)3. $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+...}}}}$

(*)4. ∞ (infinity)

Decide which of the following numbers are prime:
(*)5. 2; 5; 8; 0; 51; 1

(**)6. -17; 13689; 52841

(***)7. 89575378759; 852895895231; 3554389342359862463

Decide whether the following pairs of numbers are relatively prime:
(*)8. 2 and 3

(*)9. 21 and 36

(*)10. 216 and 144

(*)11. 121 and 169

(**)12. 40123 and 7433

(**)13. 13 and 1313131313131313131313131313...1313131313...131313

1.0.2 Questions for chapter 3

Prove the values that solve the following using the standard set of equality postulates:

(*)1. 6x+2=3x+5

(*)2. 2x+3y=2 and 2x+4y=4

(**)3. 12x+2y=6 and 15x+3z=9 and 4y+5z=-71

1.0.3 Questions for chapter 4

Find appropriate counter-examples where possible (not all are possible to find counter-examples for):

(*)1. Proposition: In the interval [0,1) there are more natural numbers than irrational numbers.

(*)2. Proposition: $x^2 + 4x + 5 = 0$ has no real values of x which satisfy the equation.

(*)3. Proposition: Everyone in your immediate family is left-handed.

(**)4. Proposition: There are no other triplet primes than 3,5,7.

(**)5. Proposition: define a perfect number as a number whose sum of all factors (prime and composite) are equal to twice its value. For example, 6 has factors $1, 2, 3, 6$, which sum to 12. We also notice that 28 is a perfect number. I propose that there are no other perfect numbers than 6 and 28.

(*****)6. Proposition: define a perfect number as above. I propose that there are no odd perfect numbers.

1.0.4 Questions for chapter 5

Prove the following using induction:
(*)1. $\Sigma 2f(i) = 2\Sigma f(i)$

(*)2. $\sum_{i=0}^{n-1}(2i+1) = n^2$

(**)3. $\sum_{i=0}^{n} i^3 = (\sum_{i=0}^{n} i)^2$

(**)4. $\sum_{i=1}^{n}(\frac{1}{2})^i = 1 - \sum_{n+1}^{\infty}(\frac{1}{2})^i$. This one is a little less formal than one might desire, as ∞ not very well defined, nor is it clear that ∞ is a natural number. This still serves as practice.

(*)5. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

These are somewhat different than the previous examples (in that they are not summation proofs):

(*)6. Prove that $8^n - 3^n$ is divisible by 5, $\forall n \geq 1$

(*)7. Prove that $n^3 - n$ is always divisible by 6, $\forall n \geq 2$. Hint: factoring might help, also it might be useful to show that this is divisible by 2 and by 3, rather than jump straight to 6

(**)8. Prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7, $\forall n \geq 1$

(**)9. Prove that $11^n - 6$ is divisible by 5, $\forall n \geq 1$

(*)10. Prove $\prod_{i=1}^{n} \frac{i}{i+1} = \frac{1}{n+1}$. [n.b.: The symbol Π is like Σ except instead of adding progressive terms, successive terms are multiplied together.]

(****)11. Attempt to prove that:

$$\frac{\sum_i a_i}{n} \geq \sqrt[n]{\prod_i a_i}$$

Where a_i is a set of positive numbers. This is true, but challenging.

(**)12. The following is a "proof" by mathematical induction that everyone has the same birthday. Find the flaw in the proof. Explain.

Property P(n): Every member of a set of n distinct people has the same birthday.

Basis of induction: Since a set of one person has only one birthday, so P(1) is true.

Inductive step: Assume P(k) is true for a positive integer k, we will show that P(k+1) is also true.

For a set $A = \{a_1, a_2, a_3, ...a_{k+1}\}$ of k+1 distinct people, consider two subsets $B = \{a_1, a_2, a_3....a_k\}$ and $C = \{a_2, a_3...a_{k+1}\}$ each with k distinct people, obtained respectively by removing the last and first person from the set A. By the inductive assumption, every member of set B has the same birthday x, and every member of set C has the same birthday y. Since the two sets B and C have $a_2, a_3....a_k$ in common, the two birthdays x and y must be the same. As a result, every member of the set A has the same birthday and we have shown P(k+1) is true based on the inductive hypothesis that P(k) is true.

1.0.5 Questions for chapter 6

(*)1. Prove $\sqrt{3}$ is irrational

(**)2. Prove the following set of statements:

the product of two positive numbers is positive
the product of two negative numbers is positive
the produce of a positive and a negative number is negative

(**)3. Prove the following statements:

the product of two even numbers is an even number
the product of two odd numbers is an odd number
the product of an even number and an odd number is an even number

(*)4. Prove that the difference between an irrational number and a rational number must be irrational.

(*)5. Prove that if a number m is rational, and the product mn is irrational, then n must be irrational.

1.0.6 Questions for chapter 7

(*)1. How many rolls of a 6-sided die are needed to ensure that a repeated roll occurs?

(*)2. How many rolls of an n-sided die are needed to ensure that a repeated roll occurs?

(**)3. How many real numbers do we need to pick between 0 and 1 so that we have an ensured repeat?